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P. L. Chebyshev (1821–1894)

A Guide to his Life and Work*

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The aim of this paper is to outline the life and work of Chebyshev, creator in St. Petersburg of the largest prerevolutionary school of mathematics in Russia, who permitted himself to be equated only with Archimedes. Chebyshev, who was regularly in Paris, at the latest by 1852, if not already by 1842, a friend of Liouville and Hermite, was the author of ca 80 publications, covering approximation theory, probability theory, number theory, theory of mechanisms, as well as many problems of analysis and practical mathematics. He was also proud to be a constructor of various mechanisms, including an arithmomètre. Although the paper is intended for an approximation theorist readership, an attempt is made to give proportionate coverage of the broad spectrum of Chebyshev's achievements, emphasis being placed upon their background. The paper is based in part upon the authors' studies during 1985–1991. © 1999 Academic Press

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1. INTRODUCTION

Pafnutii Lvovich Chebyshev[1] was born on May 16, 1821 in Okatovo, Kaluga region of Russia, on the small estate of his parents, Lev Pavlovich Chebyshev and Agrafena Ivanovna Pozniakova Chebysheva. He was one of nine children; his younger brother Vladimir, a general and professor at the St. Petersburg Artillery Academy, also is well known. His father was a retired army officer who had fought in the war against Napoleon. Recently an interesting article on the history of the Chebyshev family appeared (K. V. Chebysheva [30]) reporting that one of their ancestors was the Tartar military leader Khan Chabysh, mentioned in chronicles of the 17th century.

Chebyshev received his primary education at home; his mother taught him reading and writing. Avdotia Kvintillianova Soukhareva, apparently a cousin, fulfilled the role of governess, teaching him French and arithmetic (see [76, pp. 18, 30]).

The Chebyshev family moved in 1832 to Moscow where Pafnutii completed his secondary education at home. His tutor in mathematics was P. N. Pogorelski, the author of popular mathematical textbooks. He enrolled in the department of physics and mathematics of Moscow University in 1837, studying mathematics, in particular, under N. D. Brashman[2] and N. E. Zernov (1804–1862). Chebyshev always expressed deep respect for his teacher Brashman, to whom he attributed the greatest influence on his mathematical development.

He received his candidacy (bachelor) of mathematics degree in 1841 and his master's degree in 1846. His thesis in probability theory was defended that summer. He had passed his master's examinations already in 1843; his advisor was, again, Brashman.

Since Chebyshev found no suitable teaching job in Moscow he moved to St. Petersburg University, obtaining the *venia legendi* there and a lectureship in 1847. The associated thesis dealt with integration by means of logarithms. He received his doctorate in 1849, his thesis this time was devoted to the theory of numbers. Already in 1850 he was elected extraordinary professor of mathematics and the full professorship followed in 1860.

[1] Chebyshev's name has often been transliterated as Čebyšev in the English literature or as Tchébichef (in French) or as Tschebyscheff (or Tschebyschew, in German).

[2] Nikolai Dimitrievich Bras(c)hman(n) (* Neurausnitz (= Rousinov), Moravia, near Brno; 1796–1866), in Russia from 1824, Professor at Moscow University from 1834, founder of the Moscow Math. Society, was a mathematician whose special interests lay in mechanics, specifically hydromechanics and the principle of least action. See the Great Soviet Encyclopedia, MacMillan, New York and London, 1976, Vol. 4, p. 52; Poggendorff, Vol. 1, p. 281; A. T. Grigorian: Brashman, Nikolai Dimitrievich: in: Dictionary of Scientific Biography, ed. by C. C. Gillispie, C. Scribner's Sons, New York, Vol. 2 (1970), pp. 424–425.

He was also nominated a junior academician of the St. Petersburg Academy of Sciences with the chair of applied mathematics in 1853, as an extraordinary academician in 1856 and an ordinary academician in 1859. The chairs for pure mathematics at the Academy were then occupied by P. H. Fuss (1798–1855)—a great grandson of Euler—, M. V. Ostrogradskii (1801–1862), and V.Ya. Bunyakovskii (1804–1889).

After 35 years of teaching at St. Petersburg University in 1882 he decided to retire from his professorship but continued his research work at the Academy to the very end. He died at St. Petersburg on December 8, 1894. Although he never married, Chebyshev had a daughter whom he did not acknowledge officially, but supported financially. Later he would meet her, together with her husband, a colonel Leer, and their own daughter, at the house of his sister Nadiejda in Rudakovo [76, pp. 23–24]. The "two young and beautiful daughters" seen at Chebyshev's funeral according to Grave's autobiographical notes [25] were, presumably, Mrs. Leer and her daughter.

Chebyshev's merits were recognized early in his career. He was elected a Corresponding Member of the Société Royale des Sciences of Liège and of the Société Philomathique in 1856, of the Paris Academy of Sciences in 1860, and a Foreign Associate in 1874 (the first Russian since Peter the Great), as well as a corresponding or foreign member of the Berlin Academy of Sciences (1871), the Bologna Academy (1873), the Royal Society of London (1877), the Italian Royal Academy (1880), and the Swedish Academy of Sciences (1893).

An extended selection of Chebyshev's research was published in two volumes by A. Markoff and N. Sonin [17], while his complete works appeared in five volumes much later [18].

Chebyshev is regarded as the creator of the largest prerevolutionary school of mathematics in Russia. Its most prominent members included A. N. Korkin (1837–1908) [71], Y. V. Sohotski (=J. W. Sochozki; 1842–1927), E. I. Zolotarev (1847–1878) [70], C. A. Posse (1847–1928), A. V. Vassiliev (1853–1929) [22], A. A. Markov (1856–1922) [45], V. A. Markov (1871–1897), A. M. Lyapunov (1857–1918) [90], D. A. Grave (1863–1939) [25, 36], V. A. Steklov (1864–1926) [91], G. F. Voronoi (1868–1908), and A. N. Krylov (1863–1945) [56], I. L. Ptaszycki (1854–1912), and I. I. Ivanov (1862–1939). O. Sheynin [81] mentions further students of Chebyshev, namely the educationalists N. A. Artemiev, Latyshev, and Lermantov.[3]

[3] Several of the later students of Chebyshev are (in part) his "mathematical grandsons". Thus Zolotarev is often regarded as a student of Korkin and Somov, A. A. Markov as a student of Chebyshev and Zolotarev.

Chebyshev is the author of 80 or so publications; they span a wide area of mathematics, namely approximation theory, probability theory, number theory, theory of mechanisms, as well as many problems of analysis and practical mathematics. Many of these papers were published in major journals abroad: 17 of them in Liouville's journal, at least 10 in other French journals. Three papers appeared in Crelle's journal (Germany), and five in *Acta Mathematica* (Sweden, after 1885). Most of the remaining publications are to be found in the two journals of the St. Petersburg Academy, renowned since Euler's days at the Academy.

Chebyshev began his work in analysis while working on his master exams; then he turned to probability in his Moscow master thesis, then to integration in finite terms in his *venia legendi* thesis at St. Petersburg, then to number theory and to approximation theory, etc. We shall treat these fields successively.

The courses Chebyshev taught at St. Petersburg, roughly from 1847 to 1850, were concerned with spherical trigonometry, analytical geometry, higher algebra, elliptic functions, and practical mechanics. As an extraordinary professor he taught number theory, integral calculus, theory of interpolation and theoretical mechanics. In his capacity as an ordinary professor he continued teaching the number theory course, began teaching probability theory—a subject he taught almost continuously for 22 years—as well as the theory of definite integrals and integration of differential equations. The latter course he left to a colleague in 1875. Apart from the theory of interpolation there is no course under the specific name of approximation theory, or better still, constructive function theory, as S. N. Bernstein coined it and the Russians still call it (perhaps such lectures existed nowhere in the world at the time). However, in 1849–1851 and 1852–1856 he lectured on practical mechanics in the (short-lived) department of practical knowledge (thus quasi-enginering) of St. Petersburg University and at the Alexander Lyceum in Tsarskoe Selo (now Pushkin), respectively. It is perhaps in these lectures that he developed his original ideas on the construction of mechanisms and introduced his first personal views anticipating the constructive theory of functions.

As to his lectures and teaching, Liapunov [94], who attended Chebyshev's courses in the late 1870's, characterized them as follows:

> *His courses were not voluminous, and he did not consider the quantity of knowledge delivered; rather, he aspired to elucidate some of the most important aspects of the problems he spoke on. These were lively, absorbing lectures; curious remarks on the significance and importance of certain problems and scientific methods were always abundant. Sometimes he made a remark in passing, in connection with some concrete case they had considered, but those who attended always kept it in mind. Consequently, his lectures were highly stimulating; students received something new and essential at each lecture; he taught broader views and unusual standpoints.*

In this respect Grave (see [25]), who entered Petersburg university in 1881, added:

> *Chebyshev was a wonderful lecturer. His courses were very short. As soon as the bell sounded, he immediately dropped the chalk, and, limping, left the auditorium. On the other hand he was punctual and not late for classes. Particularly interesting were his digressions when he told us about what he had spoken outside the country or about the response of* Hermite *or others. Then the whole auditorium strained not to miss a word.*

Posse testified at the turn of the 19th century that Chebyshev's lectures enjoyed wide popularity and Lermantov stated in 1911 that Chebyshev was "a veritable teacher of mathematics" [81]. However, Prudnikov stated in 1964 [76, p. 183] without justifying his source, "It was almost impossible to take down Chebyshev's lectures in detail and, understandably, Liapunov's [...] notes are fragmentary." This was not Krylov's [55] opinion in 1936, based on lectures of his of 1880:

> Liapunov *took down* Chebyshev's *lectures with great care, each time putting his notes in order in the evening of the same day and rewriting them in his splendid hand; and since he distinguished himself not only by his knowledge but by excellent memory as well, his notes reproduce* Chebyshev's *lectures exactly as they were read, including all the fine points with which* Chebyshev *knew how to enliven his lectures in passing.*

Chebyshev was also very active as an educator. As a member of the Scientific Committee of the Ministry of Education from 1856–1873, Chebyshev was, like Lobachevskii, Ostrogradskii and other Russian scientists, active in working for the improvement of the teaching of mathematics, physics and astronomy in secondary schools, both in regard to curriculae and textbooks intended for school use. He also participated in preparing a new university charter.

As to Chebyshev himself, Grave (see [25]) writes: "Chebyshev was on the thin side, with one leg shorter than the other, and limped substantially, supporting himself with a walking stick." As to his pride Grave added that Chebyshev mentioned that, whereas Newton received the title "associé" of the Paris Academy of Sciences at the age of 57, he received it at 53. Further, he permitted himself to be equated only with Archimedes, alluding to his splendid mechanisms. He was offended when, at an international congress, someone described him as a "*splendid Russian mathematician*." He asked "*Why Russian, and not world-scale?*"

Grave also described Chebyshev's personal life. Chebyshev was a very rich man. After his death, apart from money there were a large number of estates. At one time he said to Grave: "*I don't smoke, I don't drink, and don't play cards. My only pleasure is to buy estates* [...]." Chebyshev had an official residence of 10 rooms in which he lived alone with a housemaid from whom he always locked himself up for the night, as became apparent after his death.

In this guide an attempt is made to give equal and proportionate coverage of the broad spectrum of Chebyshev's achievements. Thus his papers in the wide area of approximation theory are also only surveyed quite briefly. Let us mention, however, [53] as a good detailed survey of the part restricted to moment problems[4].

2. CHEBYSHEV'S EARLIEST WORK; HIS CONTACTS WITH LIOUVILLE

Some of Chebyshev's earliest works, in Russian, were not presented to scientific journals at the time but were published well after his death, chiefly in his collected works [17]. His first paper, written as a student in 1840/1841 and awarded a silver medal [18, Vol. 5], deals with the approximation of real roots of equations; a more refined version of the Newton–Raphson formula adds an error estimate to the latter. His thesis *pro venia legendi*, entitled "On integration by means of logarithms," which was defended in the spring of 1847, existed as a first draft as early as the end of 1843. It was also published posthumously, but a selection of its principal results appeared in 1853; Liouville had commissioned Chebyshev on Aug. 20, 1852 to publish it in *Liouville's Journal* (see below). The matter was similar in case of the master's thesis (1846) (see Section 3).

In order to obtain an international audience, Chebyshev soon realized that it was necessary to publish abroad, thus in a language other than Russian. The language was French, the place was chiefly the *Journal des mathématiques pures et appliquées*, founded by Liouville in 1836 and familiarly called *Liouville's Journal*. Chebyshev's first published paper [1], "Note sur une classe d'intégrales définies multiples," containing a formula on multiple integrals without proof and presumably presented to Liouville at the end of 1842, appeared in 1843 [29]. Surprisingly, following it in the same issue of this *Journal*, one finds a proof of the formula by Catalan (1814–1894). In his famous report of 1852 about his trip to Western Europe, Chebyshev [4] mentions rightly that he "*collaborated with this Journal since 1842*."

Now the manuscript could have reached Liouville either by mail, with an explanatory letter, by a messenger, or by a direct contact between the two in Paris. The first alternative was obviously the simplest, but no accompanying letter by Brashman or Chebyshev is recorded. In fact, according

[4] As to approximation theory, there exists the (Russian) survey by Geronimus [40].

to the huge Nachlass of Liouville at the Institut de France, complemented by his immense family archive at Bordeaux, all investigated by Neuenschwander [65, 66], there is just one letter written by Chebyshev to Liouville, of October 1873 (and dealing with one of the three last papers he published, all in 1873, in *Liouville's Journal*). Conversely, only one letter of Liouville to Chebyshev, of March 1864, has been found in Chebyshev's Nachlass in Moscow [18] or in Liouville's drafts, although Chebyshev published 17 articles in the latter's journal. In this respect it is known that Chebyshev was a notoriously bad correspondent. One exception to this strong reluctance for letter writing is known: Sophie Kovalevskaya received at least six letters from Chebyshev. Hermite was a smaller exception.

The second alternative is that Chebyshev sent his first manuscript to Liouville by some Russian travelling to Paris; a later such intermediary was N. W. Khanikov (1819–1878), a Russian geographer interested in mathematics, whose main residence was in France. According to Liouville's note books, Khanikov paid him numerous visits. But as to Chebyshev's first paper, of 1843, another messenger is more plausible; the Russian geographer Pierre de Tchihatchef (Piotr Aleksandrovich Chikhachev, (1808–1890), according to the English transliteration). This Russian, who arrived in Paris in December of 1842 in order to have his book on his journey to the Altai Mountains published in France, contacted the Sorbonne algebraicist L. B. Francoeur (1773–1849) for private lessons in trigonometry and logarithms. He in turn directed Tchihatchef to Catalan who helped Liouville in running his journal. So it was easy for Catalan to add his complement to Chebyshev's paper.

The third alternative, that Chebyshev himself brought a manuscript to Paris and there presented it to Liouville, is another possibility. In fact, Vassiliev [92] states that Chebyshev spent "*almost every summer abroad*" and Posse [75] adds that when he indeed remained in Russia for his vacation he stayed in Catherinenthal (near Reval). But neither give explicit dates when referring to his trips abroad, except for the 1852 tour. On the other hand, Chebyshev was in France at least in 1852, 1856, 1864, 1873, 1875, 1876, 1878, 1882, 1884, 1893, according to various sources.

Chebyshev, of course, also had other reasons for going to Paris; he enjoyed there the atmosphere of freedom, the scientific discussions with Liouville (who spent his summer vacations in his house and vineyards at Toul (Lorraine), where he welcomed his friends), with Hermite and others.

He also attended in France the sessions of the *Association française* in Lyon (1873), Clermont-Ferrand (1876), Paris (1878), and La Rochelle (1882). In Paris Chebyshev stayed repeatedly in the *Hôtel Corneille*, opposite l'Odéon.

For none of Chebyshev's papers in *Liouville's Journal* do we know in which of the three alternatives they reached Liouville. Even for the first paper it is theoretically possible that Chebyshev presented it to Liouville; Tchichatchef perhaps took the twenty-one year old Chebyshev with him to Paris as a "secretary," introduced him (via Francoeur) to Catalan, and so to Liouville. For this material see [28].

Chebyshev's second and third printed papers, both in French, devoted respectively to the convergence of Taylor series and probability theory, appeared in Crelle's *Journal für Reine und Angewandte Mathematik* (1844, 1846) [2, 3]. Again it is not known in what fashion Chebyshev communicated these papers to Crelle; in any case no contacts between the two have been found in German sources.

Chebyshev published only once more in Crelle's journal, namely in 1855, the year of Crelle's death. This is surprising in view of his friendly contacts with Borchardt, who followed Crelle as the editor. So Chebyshev's chief medium for his mathematical achievements in Western Europe, remained, for 30 years, *Liouville's Journal*. However, when Liouville transmitted the editorship of his journal to H. Resal (1875), then Chebyshev published in other French or Belgian journals, chiefly in the newly founded *Bulletin de la Société mathématique de France*.

As Chebyshev's reputation in the West became well established, his publishing conditions changed gradually: he published more and more in Russia, chiefly in the bulletins and memoirs of St. Petersburg Academy. First these papers were in French, which allowed a second printing in France; later they were all in Russian (several were translated into French by Mention, Bienaymé, Khanikov, Dwelshauvers–Déry, Falisse, Cuyper, Kowalewskaja, Lyon, and one into German by Backlund).

3. CHEBYSHEV'S WORK IN PROBABILITY THEORY

His work in this field began with his master's thesis (1846), published well after his death in Vol. V of [18]. An accompanying paper appeared, however, in 1846 in Crelle's journal, under the title "Démonstration élémentaire d'une proposition générale de la théorie des probabilités" [3]. It contained an analytic approach to Poisson's weak law of large numbers, stating that the number X of successes in n independent trials is related to the probabilities p_i of success in the ith trial via the arithmetic mean $\bar{p}(n)$ of the p_i's by

$$\lim_{n \to \infty} Pr[\,|(X/n) - \bar{p}(n)| < \varepsilon] = 1$$

for any positive ε. Remaining unnoticed at the time, this paper had thus no effect on the controversy about laws of large numbers then going on in France.

In a paper on continued fractions, first published in Russian (1855), afterwards translated into French by Bienaymé for *Liouville's Journal* (1858) [9] and substantially devoted to a system of "orthogonal" polynomials which now bear his name, Chebyshev did not contribute directly to probability theory nor to statistics, but Bienaymé's comments about it induced him to return to the topic in 1859 [12] with a treatment delving deeper into the problem of fitting a polynomial to n pairs of observations (values of x associated with the values of the polynomial) by a more elaborate use of orthogonalization. This introduced a new point of view in a problem of mathematical statistics already solved by Cauchy in 1853. Chebyshev's main application of the powerful theory of orthogonal polynomials is, however, to be found in the rich field of approximation theory (see Section 5).

Perhaps Chebyshev's most widely known contribution to probability theory is the so-called *Bienaymé–Chebyshev inequality*, stating that the probability for a random variable X to differ from its mean μ by no more than $t\sigma$, σ being the standard deviation of X and t any positive number, is at least $1-t^{-2}$. Although he obtained this inequality in 1867 [13], 14 years after Bienaymé and in a more restrictive setting, a fact which he admitted in 1874, the inequality remained popular in Russia under Chebyshev's name alone, together with the substantial complement (also obtained by Bienaymé) of a weak law of large numbers. This application of the inequality covered both cases previously treated by Poisson and J. Bernoulli.

Finally in 1891 Chebyshev published, in *Acta Mathematica* [16], a paper which had previously appeared in Russian (in 1887), attempting to prove via the "method of moments" the central limit theorem for the sum of (independent) not identically distributed summands. The gaps in assumptions and proof provoked much discussion in Russia until A. A. Markov overcame the inadequacies using Chebyshev's method in 1898, and Liapunov deduced a little later (1901, [58]) a very general version with a proof using characteristic functions. However, a rigorous proof in a more restrictive setting had already been sketched by Cauchy in 1853 and completed by I. V. Sleshinsky [82] of Odessa in 1892.

Chebyshev taught probability theory regularly from 1860 to 1882, apparently more than his earlier courses on higher algebra and number theory. Recently, O. Sheynin [81] published a description of the lectures on probability that Chebyshev held at St. Petersburg in 1879–1880. It is based on lecture notes written down by A. M. Liapunov and published by A. N. Krylov [55] in 1936. According to N. Ermolaeva (1986) (see

[81, p. 323]), there exist much more detailed, still unpublished lecture notes taken during Sept. 1876–March 1878, possibly by N. A. Artemiev.

In short, Chebyshev's production in probability theory was not voluminous, but it was ground-breaking. He derived the law of large numbers in a general setting, played his part in discovering the Bienaymé–Chebyshev inequality, and he attained serious success in the battle about the central limit theorem. His work gave a strong impulse to the Russian probabilistic school, in particular to the highlights of his great disciples A. A. Markov and Liapunov. But to the English-speaking world the significance of these contributions was not immediately apparent, judging by an obituary address [20] of 1895. Even early papers by Shleshinsky [82] and P. A. Nekrasov (1853–1894) [64] remained, if not unnoticed, at least underestimated for quite a long time. The work of E. Seneta [79, 80] inspired this section. See also [43, 47].

4. CHEBYSHEV'S WORK ON INTEGRATION IN FINITE TERMS

As to his above area of research Chebyshev had been asked by Liouville to publish the principal results of his thesis *pro venia legendi*, as already mentioned, a fact which Chebyshev confirms in his report [4, p. XV], adding that "Liouville and Hermite suggested the idea to develop the principles on which my thesis had been based." Further, "in this paper [thesis] I considered the case where the differential under the integral contains the square root of a rational function. But it was interesting in several respects to extend those principles to a root of any degree," as suggested by the two of them.

In this thesis Chebyshev established a conjecture of Abel of 1826 to the effect that if the integral $\int (\varrho(x)/\sqrt{R(x)})\,dx$, ϱ and R being polynomials, is expressible by logarithms, then it can be written in the form

$$\int \frac{\varrho(x)}{\sqrt{R(x)}}\,dx = c \log \frac{p+q\sqrt{R}}{p-q\sqrt{R}},$$

where p and q are entire functions and c is a constant. Upon the suggestion of Liouville and Hermite, Chebyshev in 1852 [17, Vol. I, pp. 147–168]) considered more generally the integral $\int (f(x)/\sqrt[m]{R(x)})\,dx$, where f is only supposed to be rational but R is still a polynomial. Now according to a result of Liouville and Abel, if this integral is expressible in finite form, it has to be of the form

$$U + c_0 \log V_0 + c_1 \log V_1 + \cdots + c_n \log V_n,$$

where U, V_0, V_1, .., V_n are rational functions of $x + \sqrt[m]{R(x)}$, the c_i being constants. Chebyshev himself first showed how to determine the algebraic part U, thereby generalizing Ostrogradskii's method, and then he determined how many terms of the form $c_i \log V_i$ are needed. In particular, $\int (\varrho(x)/\sqrt[m]{R(x)})\,dx$ is not expressible in finite form if $R(x)$ has no roots of multiplicity greater than m and ϱ is a polynomial of degree less than the degree of $\sqrt[m]{R(x)}/x$. He also showed how to reduce the general problem to that of deciding the integrability of $\int ((x+c)/\sqrt{R(x)})\,dx$ in logarithms. Weierstrass who, together with Chebyshev, became the second co-founder of approximation theory (with the theorem named after him of 1885), already criticized Chebyshev's methods of 1857 in that same year; he preferred to solve the problem using Jacobi's theory of elliptic functions, which gave a "clearer and deeper insight into the essence of the matter" [61, p. 417].

Chebyshev wrote five further papers on the subject (appearing in 1857 [8], 1860 [17, Vol. I, pp. 509–517], 1861 [17, Vol. I, pp. 514–530], 1865 [17, Vol. I, pp. 563–608] and 1867 [17, Vol. II, pp. 43–47]) and it was followed up by Zolotarev in 1874 [96]. Whereas Dirichlet had shown vivid interest in Liouville's work (see his letter of May 6, 1840 to Liouville in [61, p. 254], Chebyshev was the first to become actively engaged in it. J. F. Ritt [77] published a comprehensive book on integration in finite terms.

According to Youshkevich [94, p. 223], Chebyshev's thesis solved a problem "posed shortly before by Ostrogradskii", but Youshkevich gives neither dates nor references. However Chebyshev, both in the French version of 1853 of his thesis and in the original Russian version, refers only to work of Abel and Liouville in the matter; also in his paper of 1857 [8] he does not cite Ostrogradskii, but only Abel, Liouville and Hermite. In fact, one of Liouville's chief mathematical interests between the years 1833 and 1841 was the field of integration in finite terms in which he continued work begun by Abel in 1823, as can be deduced from the oustanding work by Jesper Lützen [61] on Liouville. Since Ostrogradskii's first paper on integration of rational functions, presented to the St. Petersburg Academy on Nov. 22, 1844, appeared in 1845 (see [61, pp. 262, 414]), but Chebyshev's first draft was completed by the end of 1843, one may question the claim that Chebyshev's thesis was influenced by Ostrogradskii. In any case the latter was in far-off St. Petersburg; Chebyshev was still in Moscow at the time. However, it is possible that Chebyshev's work here was also influenced by that of Brashman and O. I. Somov (1815–1876) [67] who in turn were perhaps stimulated by the publications of Abel and Liouville. Moreover, an important memoir by Condorcet [42] of 1775 was largely overlooked until recently.

5. CHEBYSHEV'S WORK IN APPROXIMATION

5.1. *Background Material*

Chebyshev's first work on approximation, namely his paper "Théorie des mécanismes connus sous le nom de parallélogrammes" [7], was read to the St. Petersburg Academy on January 28, 1853 and published in 1854. This work, followed by "Sur les questions de minima qui se rattachent à la représentation approximative des fonctions" [10], read October 9, 1857, but published in 1859, marked the beginning of his 40-year research on approximation and the study of mechanisms. Thematically, Chebyshev's total work on approximation included the theory of orthogonal polynomials, interpolation, theory of moments, integration, approximate quadratures, and continued fractions.

Let us first consider the background to Chebyshev's work on approximation. It is his interest in the theory of mechanisms in the field of applied mechanics that resulted in his theory of best approximation of functions. This interest was enhanced not only by that of his teacher Brashman but most probably also by that of the two French scientists Gabriel Lamé (1795–1870) and Benoît-Pierre-Emile Clapeyron (1799–1864) who taught during their "exile" (1820–1831) at St. Petersburg Institute of Ways of Communications. Both were noted for their outstanding contributions to the development of mathematics, mechanics, applied physics, and the art of constructions. It was also at this Institute, Russia's first technical school noted for construction mechanics and civil engineering, that Chebyshev's colleague Ostrogradskii taught from 1830 on. Chebyshev's interest in applied mechanics is also documented by the series of lectures he presented on the subject between 1849 and 1856, despite their elementary level.

But the major stimulus to Chebyshev's work in approximation, at least, was his grand tour to France, England, and Germany from July to November 1852 (see his detailed report in [4]). It took him in France to the Conservatoire des Arts et Métiers in Paris, the railway between Paris and St. Germain, the mines and foundries of Lorraine, the paper mills of Lille, the munition factories of Châtellerault. On the mathematical side, his contacts in Paris included Liouville, Bienaymé, Hermite, J. A. Serret (1819–1885), V. A. Lebesgue (1791–1875), and J. V. Poncelet (1788–1867),[5] and in Metz, C. A. J. de Polignac (1832–1913).

While in France, before crossing the Channel for England in October 1852, several mathematicians asked him to forward letters to J. J. Sylvester (1814–1897), who had been in France half a year earlier (see [51]). So it

[5] It is not clear whether Chebyshev actually met Poncelet.

is not surprising that Chebyshev in his report first of all mentions his visit with Sylvester and the latter's alter ego A. Cayley (1821–1895). Through them he was introducted to the "célèbre ingénieur-mécanicien" C. H. Gregory (1817–1898) [28], who became his "cicerone" in various factories in London and surroundings. In London these were Maudsley Son and Field, D. Napier and Sons, and John Penn and Sons, all builders of large steam engines. Especially the machines built by James Watt (1736–1819) stirred his imagination. In fact, the only work that Chebyshev explicitly cites in his first paper on approximation is that of Poncelet on practical mechanics (with no explicit reference, however[6]) and Watt's parallelogram. In particular, Watt's steam engine first led him to construct a linkage which converts circular to straight line motion with less discrepancy than that of Watt, and finally led him to new problems in approximation, as Chebyshev [4] asserts.

On the way back to Russia, Chebyshev stayed three days in Brussels where he visited an important museum of machines and attended a lecture on applied mechanics by Kent; this stay in Brussels was thus devoted entirely to practical mechanics.

The fact that Chebyshev was back in St. Petersburg on November 7 (thus, Nov. 19 of the Gregorian calendar), his paper [7] being presented to the Academy less than 3 months later, suggests, and Youshkevich confirms, that it was mainly written during the 1852 trip. Further, Watt and Poncelet take up about $2\frac{1}{2}$ pages of his 12 page report of his trip [4].

The tour ended in Berlin, where Chebyshev visited P. G. Lejeune Dirichlet (1805–1859), Germany's most renowned mathematician at the time. In a half page devoted to Dirichlet, he writes that "it was of great interest for me to become acquainted with the celebrated geometer Lejeune–Dirichlet," that the most important of this "savant's" investigations were the applications of infinitesimal calculus to number theory, that he himself "found an occasion each day to talk with this geometer concerning this research as well as other questions on pure and applied analysis," and that he attended "with particular pleasure one of his lectures on theoretical mechanics". He further regretted deeply that he already had to leave Berlin on October 30, due to the unexpected setting in of ice in the Gulf of Finland.

Furthermore, Chebyshev states explicitly that he talked to Dirichlet about his investigations, of 1848–1852, concerning the distribution of prime numbers [5, 6].

5.2. *The Work Itself*

In his first two cited papers concerned with approximation he raised and studied the problem of best uniform approximation of a

[6] The paper is actually [74].

function $f \in C[a, b]$ by algebraic polynomials $p_n(x)$ of degree $\leqslant n$, defining

$$E_n[f] = \inf_{p_n \in \mathscr{P}_n} \{ \max_{x \in [a, b]} |f(x) - p_n(x)| \},$$

called the best approximation of degree n of f, $\mathscr{P}_n$ being the set of all $p_n(x)$. A polynomial $p_n^*(x)$ of best approximation to $f \in C[a, b]$, defined by $E_n[f] = \|f(x) - p_n^*(x)\|_C$ for which the infimum is thus attained, was assumed to exist by Chebyshev. The actual existence of $p_n^*(x)$ was first established in the basic doctoral dissertation by Kirchberger (1902) and also by E. Borel(1905) [27]. Chebyshev's emphasis was on the characterization of the polynomial of the best approximation and its calculation in several important special cases, using the mathematical techniques of the time. In this respect, his alternation theorem,[7] later stated by him in a more general (but rough) form, namely for the best approximation of $f \in C[a, b]$ by rational functions with fixed degrees of the numerator and denominator, states in an exact form for polynomials that $p_n(x) \in \mathscr{P}_n$ is a polynomial of the best approximation to f if there exist $n+2$ points, $a \leqslant x_0 < \cdots < x_{n+1} \leqslant b$, such that $\pm(-1)^i [f(x_i) - p_n(x_i)] = \|f(x) - p_n(x)\|_C$ for $i = 0$, 1, ..., $n+1$; i.e., $f(x) - p_n(x)$ takes on the values $\pm \|f - p_n\|_C$ in alternating succession at x_i.

Note that as a consequence of the alternation theorem $p_n^*(x)$ is uniquely determined. As an example, the polynomial $p_n^*(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ of the best approximation to $f(x) = 0$ on the interval $[-1, 1]$ turns out to be the Chebyshev polynomial (with leading coefficient one) $2^{1-n} T_n(x) = 2^{1-n} \cos(n \operatorname{arc\,cos} x)$, $n \geqslant 1$, with $\|2^{1-n} T_n(x)\|_C = 2^{1-n}$. Thus the maximum deviation of this polynomial from zero is 2^{1-n}. The $T_n(x)$ form an orthogonal system with respect to the weight function $(1 - x^2)^{-1/2}$.

[7] The first proof of the alternation theorem is also due to Kirchberger (1902); a last gap was filled by J. W. Young (1907), who can thus be credited with the first, fully correct proof. As Steffens [84, p. 78] argues, Kirchberger's proof, which depended on results by Chebyshev, reveals how close Chebyshev himself was to a proof of the alternation theorem; the results had to be interpreted correctly. After the present paper was written, the authors received, upon the suggestion of a referee, an informative paper on the alternation theorem, by K.-G. Steffens [84], a student of Bruno Brosowski at Frankfurt/Main, who spent eleven months in St. Petersbourg. In fact, he writes (p. 65) that in Chebyshev's work one finds no real proof of the alternation theorem, as also G. I. Natanson and N. S. Ermolaeva (both St. Petersbourg) have confirmed. More concretely, according to V. L. Goncharov [44, p. 146], in Chebyshev's total collected works, "there is no hint at all as to an alternation of the signs of the values of the alternating points." "But this does not mean that the alternation property was unknown to him," A. A. Gusak (Minsk, 1972) is, however, not as sharp in his evaluation.

In a second long memoir [10] Chebyshev also extended the alternation[8] problem to all kinds of functions $F(x, a_1, a_2, ..., a_n)$ depending on n parameters, expressed general views concerning the method of solution, and gave a complete analysis of two cases when F is a rational function with fixed degrees of the numerator and denominator.

Chebyshev's first paper on the general theory of orthogonal polynomials on a finite set of points seems to be his paper [9] of 1855, mentioned briefly in Section 3. It contains the three term recurrence formulae for orthogonal polynomials (considered by Christoffel in 1858; see [39]), the Christoffel–Darboux formula (considered for Legendre polynomials by Christoffel in 1858), and the fact that if a matrix is orthogonal, then its transpose is also orthogonal. Bienaymé, the translator of this first paper, calls attention to a two page note of Chebyshev (see [17, pp. 701–702]) which introduced a set of polynomials orthogonal with respect to the uniform distribution on a finite set of equally spaced points. Chebyshev observed that these generalize Legendre polynomials. This note also mentions the general case of an arbitrary set of finite points, and the connection with least squares and with continued fractions.

Then in two short papers of 1856 and 1858 (see [17, pp. 231–236, 379–384]) Chebyshev found the polynomials orthogonal with respect to the uniform distribution on the points 1, 2, ..., n and pointed out that they are a discrete analogue of the Legendre polynomials. In his paper of 1859 [11], instead of taking up the determination of polynomials under the conditions of orthogonality in a given interval with respect to a given weight function, Chebyshev's basic research apparatus was the expansion

[8] According to Cheney's excellent notes on the history of approximation in [31, p. 327], the alternation theorem for polynomial approximation was first proved by Borel (1905) [27], although it is often incorrectly attributed to Chebyshev. This is indeed so in the Russian literature. Although Natanson [63, p. 24] cites Borel's tract, A. F. Timan [89] does not. In a personal communication of Nov. 5, 1996 Ward Cheney mentions that Chebyshev did not furnish a complete proof because the standards of his day did not permit it; the compactness and continuity arguments that a proof required were not understood at the time. Vladimir Tikhomirov, who lectured in Aachen on Nov. 18th, 1996 confirmed this view. In fact, he recalled that Chebyshev's remarks in [7] in regard to the alternation theorem give the impression that it was already known at the time. Indeed he writes [7], last lines of p. 114 "as one knows (=comme on le sait) ... ." In regard to Chebyshev's actual statement of the theorem there are two details. First, he asserts (in other words) that the maxima and minima of $f(x) - p_n(x)$ are reached *at least* $n+2$ times and have (implicitly) the same absolute value; the endpoints a, b of the interval are *always* among these points; this is perhaps an "approximate" way of asserting it. However, on p. 115 (lines 2, 9 and even 13) he speaks twice of the $n+2$ maxima and minima of $f(x) - p_n(x)$. On the whole it is not quite clear whether the extrema of $f(x) - p_n(x)$ in alternating succession, including the two endpoints, have cardinality $n+2$ or perhaps higher. Thus Chebyshev gives an "approximate" version, without proof, of the alternation theorem. Since this theorem was known according to Chebyshev, no literature being given, the question arises whether it is possibly a result of Poncelet's work.

in *continued fractions* of the integral $\int_a^b w(u)(x-u)^{-1}\,du$; the dominators of the convergents of this continued fraction form a system of orthogonal polynomials on the interval (a, b) with weight $w(x)$. In this respect he studied again the Chebyshev polynomials and rediscovered the Hermite and Laguerre polynomials; he also wrote down the first four of each. The Hermite polynomials $H_n(x)$ are orthogonal on $(-\infty, \infty)$ with weight function $\exp(-x^2)$, defined by the Rodrigues formula $H_n(x)=(-1)^n \exp(x^2)\cdot(\exp(-x^2))^{(n)}$; the Laguerre polynomials $L_n(x)$ are orthogonal on $(0, \infty)$ with weight $\exp(-x)$, defined by $L_n(x)=(e^x/n!)(e^{-x}x^n)^{(n)}$.

His final paper dealing with orthogonal polynomials followed in 1875 [15]. There he found what are now called Hahn polynomials which are discrete extensions of Jacobi polynomials, and a discrete version of Rodrigues' formula for them; he also gave their orthogonality relation.

Historically, the first orthogonal polynomials were the Legendre polynomials (of 1785, a recurrence relation for them was already found by Laguerre), to be followed by the Chebyshev–Hermite polynomials (of 1859, 1864 by Hermite, especially in several variables) but already used by Laplace in his work of probability (of 1810). As to the Laguerre polynomials, a first published work on them using their orthogonality, was by R. Murphy (1833–1835); Abel's earlier work on them was published posthumously in 1881. As to continued fractions, there is important work by Gauss, Christoffel, Mehler, and Heine. See R. Askey's account in [87].

Preceding the fundamental work of T. J. Stieltjes (1894) on the moment problem in the real domain, there was the less general and less precise work of Chebyshev between 1871 and 1882 [17, Vol. II, pp. 107–126, 299–331, 333–356, 357–374] (and later A. A. Markov of 1884/1896). They described the properties of a class U of functions defined on $(-\infty, \infty)$ such that the relations $f(x)\in U$ and

$$\int_{-\infty}^{\infty} x^n f(x)\,dx=\int_{-\infty}^{\infty} x^n \exp(-x^2)\,dx, \qquad n\in N_0 \tag{$*$}$$

lead to the identity $f(x)=\exp(-x^2)$. The question here concerns a maximally complete and constructive characterization of the uniqueness class U of the interpolation problem $(*)$. The solution of the moment problem $(*)$ plays a major role in probability theory and statistics. Chebyshev's main tool again was the theory of continued fractions.

In Chebyshev's quadrature formula of 1873/1874 [14],

$$\int_{-1}^{1} f(x)\,dx\approx c\sum_{k=1}^{N} f(x_k), \tag{$**$}$$

with equal coefficients and weight function 1 (a particular case of the Gauss–Chebyshev formula with weight function 1), the parameters

$x_1, ..., x_N$ are determined by the requirement that (**) be exact for all polynomials $p_N(x)$, and c by relation (**) being exact for $f(x) = 1$, with value $2/N$. For $N = 1, 2, ..., 7$ the nodes x_k were calculated by Chebyshev. They are real only for these N as well as for $N = 9$. According to S. Bernstein (1937), the nodes are all complex for $N = 8$ and $N \geqslant 10$ so that (**) is unusable then. See, e.g., [46] as well as Vol. 7, pp. 387–390 there.

6. NUMBER THEORY

Shortly after his arrival at St. Petersburg, Chebyshev became a collaborator of Bunyakovskii for a complete edition of Euler's 99 papers in number theory; he contributed in particular to the remarkable "Index systématique et raisonné" listing their bibliographical research (1849). This enterprise may have stimulated his interest in the field although he had already submitted for his doctoral thesis a monograph on the theory of congruences; this was later translated from Russian into German (1888) and Italian (1895). But the main problem treated by Chebyshev in number theory was the distribution law of prime numbers among all integers, a problem to which he devoted two important papers in the "Mémoires des savants étrangers" (1848, 1850) of the St. Petersburg Academy; a second publication of both appeared in *Liouville's Journal* of 1852 [5, 6]. In the first paper the number $\pi(x)$ of primes not exceeding x (>2) is shown to differ from $\int_2^x du/\ln u$ by less than $ax/\ln^b x$ for infinitely many x, however small a (>0) and however large b may be. From this it follows that $x/\pi(x) - \ln x$ cannot have a limit other than -1 for $x \to \infty$, whereas an empirical approximation of $\pi(x)$ by Legendre tended to suggest that this same expression had the limit -1.08366. In fact Legendre's approximation conformed satisfactorily with the table of prime numbers up to one million, but not beyond it. On the contrary, $\int_2^x du/\ln u$ may be considered as a good approximation of $\pi(x)$ for a large x, a law in accordance with an empirical statement by Gauss.

In his second mentioned paper Chebyshev proves, by a fairly long reasoning, a conjecture of Bertrand: for any integer $x > 3$ there exists some prime number strictly larger than x and smaller than $2x - 2$. This is a particular consequence of the fact that there are more than k primes between l and L when

$$l = \frac{5}{6} L - 2L^{1/2} - 25 \frac{\ln^2 L}{16A \ln b} - \frac{5}{6A} \left(\frac{25}{4} + k \right) \ln L - \frac{25}{6A},$$

with $A = \ln(2^{1/2} 3^{1/9} 5^{1/6} / 30^{1/30}) \cong 0.92129$.

At the end of this paper Chebyshev comes back to the number $\pi(x)$, with precise but complicated expressions for lower and upper bounds of $\pi(x)$. The main terms of these bounds are respectively $Ax/\ln x \cong 0.92129x/\ln x$ and $6/5Ax/\ln x \cong 1.10555x/\ln x$, in such a way that, for sufficiently large x, the ratio of $\pi(x)$ to $x/\ln x$ remains between 0.92129 and 1.10555, thus close to 1, and even closer as Sylvester showed in 1891. Chebyshev did not actually present $x/\ln x$ as a decisive approximation of $\pi(x)$, but this is an implicit consequence of his formulas. In 1896, shortly after his death, this approximation was explicitly and independently announced by Hadamard and La Vallée-Poussin, with proofs of the striking property that $\lim_{x\to\infty}(\pi(x)/x/\ln x)=1$. Among further contributions, for which we refer to Schwarz [78], A. Selberg and P. Erdős gave in 1949 an "elementary" proof of the prime number theorem.

7. MECHANISMS

Forced by his limping to avoid most children's games, the boy Pafnutii Lvovich began, with great love and skill, to build a number of small mechanical apparatuses. Up to his very last days, his most constant center of interest remained a wide variety of mechanisms (see, e.g., the list of eleven such in Wassilief and Delaunay [92, p. 45]). We have already seen how this serious hobby inspired his mathematical views about approximation of functions. But the technical point of view also deserves some attention. The rapid expansion of railways all over Europe after 1830 justified a special interest in steam engines; for the control of these two particular mechanisms were associated with the name of James Watt: the regulator and the so-called Watt's parallelogram. Chebyshev's first criticism towards this and other "parallelograms" (although this name is not very convenient), intended to convert a circular motion into a rectilinear one, was that they furnished a rather poor approximation of rectilinear trajectories. A second criticism still remained even after theoretically perfect "inversors," giving a precise rectilinear motion, were found later in the century by H. Hart, C. N. Peaucellier (1832–1913) and Chebyshev's student Lippman Lipkin (1851–1875) (cf. [34]). In fact, too many rigid parts (seven in the case of Peaucellier and Lipkin) and consequently the hinges joining these parts made the construction difficult and the use, under severe conditions, unsafe. Chebyshev found, among several devices, an especially simple one with just three rigid parts and four hinges, in which a vertex of a triangular rigid part described a type of a twice–twisted eight, with two nodes instead of one, squeezed in a very narrow strip between two lines parallel to the line joining the nodes. The width of this strip could be precisely controlled by one strategic variable, the angle of the triangle at a second vertex [17, Vol. II, pp. 493–540]. In a similar way Chebyshev was able to imitate the pace of a human being; he actually constructed a kind of stepping engine, instead of a car running on wheels.

Not only did Chebyshev describe his mechanisms in a series of papers with great care for practical details, but he constructed them, sometimes himself, sometimes with the help of the best craftsmen he could find in Russia or Paris. In 1874, for example, a regulator for a steam engine was made for him in Paris under the control of the engineer Eichens; it did not function properly until Khanikof, present in Paris, discovered that in the drawings made by Chebyshev and him, an angle supposed to equal 60^0 exceeded 68^0. Here Khanikof acted not only as a transmitter of information between Russia and France but as a kind of factotum [76, p. 207].

The most elaborate mechanism conceived by Chebyshev was his "arithmomètre," a calculating machine "à mouvement continu," applying another of his principles; whenever possible, continuously turning parts are preferable to those rotating alternatively or stopping instantly many times. The underlying ideas were explained in 1876 at the Clermont-Ferrand session of the Association française pour l'avancement des sciences. The construction itself, by the firm Gautier in Paris, lasted till 1882 when Chebyshev presented the calculator at the La Rochelle session of the *Association* that August. Seemingly, some further improvements took place afterwards for, according to two letters addressed to Catalan by Bunyakowski (who played for Catalan the role Khanikof did for Liouville; the dates are 15.I.1884 and 21.V.1884), the execution of the arithmetic machine was progressing satisfactorily and Chebyshev would consequently be back in Western Europe during the fall of 1884 in order to demonstrate his machine. In fact, on Dec. 7, 1884, Chebyshev presided over the ceremony in honour of the retirement of Catalan at Liège university. Most likely the European tour took place then and ended in Paris. At any rate, we know that Chebyshev made a temporary gift of his arithmometer to his friend Edouard Lucas (1842–1891), a number theorist [60 p. 74], probably shortly after this Liège visit. This gift lasted for years, presumably since Chebyshev did not return to Paris before Lucas' death.

Other models, drawings or photographs of mechanisms due to Chebyshev were exhibited by Lucas in a special showcase at the Conservatoire [76, p. 240–241]. The head of the Conservatoire, Aimé Laussedat (1819–1907), the "father of photogrammetry," another correspondent of Chebyshev, gave Maurice d'Ocagne (1862–1938) the task of writing that part of the catalogue concerning arithmetic machines. As observed in Section 2, the contacts he had with Chebyshev induced the latter to come once again to Paris for a whole month so that d'Ocagne could complete an extended description of the "arithmomètre" [68]. On

June 3, 1893, Chebyshev visited the Conservatoire and confirmed the gift of the epicycloidal train, the essential part of his machine.[9] This did not signify the end of Chebyshev's inclinations for mechanisms. At the World's Columbian Exposition held in Chicago in 1893, seven of Chebyshev's mechanisms were exhibited.

Of particular interest was a bicycle for women, as the engineer Victor Dwelshauvers–Déry (1836–1913), a professor at Liège, specialist in steam engines, and friend of Chebyshev, pointed out in a letter to the latter. He also wished that Chebyshev would expose the same mechanisms at Antwerp the following year. We do not know whether this wish was fulfilled, 1894 being, moreover the year of Chebyshev's death.

8. ON CUTTING CLOTH

In 1878, the seventh session of the Association française pour l'avancement des sciences took place in Paris. There Chebyshev presented four comunications, in particular one entitled "Sur la coupe des vêtements," the idea of which goes back to a talk given two years earlier by E. Lucas at the session of Clermont-Ferrand. Entirely written in French, Chebyshev's talk was delivered on August 28, but, as usual, only a short account appeared in the proceedings of the session. Chebyshev renounced publishing the full text elsewhere, as he sometimes did in similar cases. Nevertheless, several authors such as A. Voss, G. Darboux and L. Bianchi soon published results about networks designed on curved surfaces by initially plane and orthogonal systems of inextensible fibers; such networks were eventually called Chebyshev nets.

Arguing that the manuscript found in Chebyshev's papers after his death did not bear the word "imprimer" in the margin, Markoff and Sonin did not reproduce it in Vol. II of [17], but gave only a short comment similar to the account which appeared in the proceedings of the seventh session. A reverse choice was made in [18, Vol. 5, pp. 165–170], where we find a full Russian translation of the manuscript. An additional translation from Russian into English, by M. Chobot and B. Collomb [33], appeared in 1970. Chebyshev's vow was respected, if one dare say, in the sense that the original version in French was never printed!

Let us give a brief comment of the English version. Chebyshev's aim is the tight covering of a body of any shape by some sort of cloth, assembling warp threads and woof threads supposed to be inextensible, in such a way that the bending of this material in order to coat the object alters only the

[9] Another specimen of the machine was constructed after Chebyshev's death for exhibition in Moscow (see [69, Vol. 1, p. 26]).

original right angle between warp and woof. The length x of warp and y of woof being taken in an obvious way as curvilinear coordinates of any point of the extended material from a given origin, the original elementary distance $dx^2 + dy^2$ in the plane becomes $ds^2 = dx^2 + dy^2 + 2\cos\phi\, dx\, dy$ if the angle between the threads passing in (x, y) has been changed from $\pi/2$ into ϕ (varying from point to point). The geodesics associated with this ds^2 may be obtained, thanks to the calculus of variation, by making the variation of $\int ds$ equal to zero. This leads to a development of $\cos\phi$ as a power series in x, y, and ultimately to the expression of x, y as power series expansions in s, the first three terms only being taken. From there on, Chebyshev asserts without much detail the possibility of finding the (geodesic) curves along which the pieces of material should be cut in order to obtain a partial coating of the given body, the various parts to be coated separately having been chosen in advance. Those "border conditions" remain quite vague, but as an example Chebyshev presented in Paris a two-piece coating of a spherical ball; each piece was a four-sided stellar figure with curvilinear "prickles," the diagonals of which correspond to the x and y axis. Each piece is bent over an hemisphere of the ball; the common border is a great circle, i.e. a geodesic of the sphere. This model differs obviously from the now traditional coating of a tennis ball.

On the whole, starting with a fine idea, Chebyshev's paper gives in a sketchy way a purely geometric approach of what is ultimately an equilibrium problem of a network subject to suitable boundary tractions. Accordingly, further developments of Chebyshev's basic idea were mainly published in journals concerned with mechanics or engineering. The theory of plane networks of inextensible chords and their deformations was successfully studied by R. S. Rivlin between 1955 and 1964. From 1980 on, A. C. Pipkin and C. G. Rogers made further progress in the more general case of three–dimensional deformations, raised by Chebyshev one century earlier; for precise references see Pipkin [73], a paper significantly entitled Equilibrium of Tchebyshev Nets.

9. AN ASSESSMENT OF CHEBYSHEV'S WORK

As to Chebyshev's mathematical contributions in general, A. V. Wassiliev [92, p. 48] reported that Chebyshev said to him some years before his death, perhaps half seriously:

> Mathematics has already gone through two periods: during the first the problems were posed by the Gods (the Delos problem of the duplication of a cube) and during the second by demigods, such as Fermat, Pascal, and others. We now enter the third period, where the needs of Mankind raise problems that must be solved.

Chebyshev expressed similar thoughts already in 1856 in a speech entitled "Drawing Geographical Maps." Indeed:

> *The closer, mutual approximation of the points of view of theory and practice brings most beneficial results, and it is not exclusively the practical side that gains; under its influence the sciences are developing in that this approximation delivers new objects of study or new aspects in subjects long familiar. In spite of the great advance of the mathematical sciences due to the works of the outstanding geometers [i.e. mathematicians] of the last three centuries, practice clearly reveals their imperfection in many respects; it suggests problems essentially new for science and thus challenges one to seek quite new methods. And if theory gains much when new applications or developments of old methods occur, the gain is still greater when new methods are discovered; and here science finds a reliable guide in practice.*

In regard to Chebyshev, the great Russian mathematician S. N. Bernstein (1880–1968), successively Professor in Kharkov, Leningrad and Moscow, already stated in 1913 in the public defense of his doctoral dissertation (see [23]):

> *The questions asked by* Chebyshev *later drew the attention of many prominent mathematicians, but none of these contributed to this field as many new and original ideas as did the originating genius himself. In none of* Chebyshev's *works on best approximation, however, nor in any of his applications do we find any indication that the great Russian mathematician was interested in the fundamental problem of whether or not it is possible to make the error arbitrarily small for any continuous function by increasing the degree of the approximating polynomials. Credit for having answered this profoundly important question [...] must go to another famous mathematician,* Weierstrass, *[... His] discovery [...] directed the investigation of approximation of functions along a new path. Whereas the path directly indicated by* Chebyshev *may be characterized rather accurately by the term algebraic, the path which arose under the influence of* Weierstrass *should properly be called analytic [...]. The basic problem [in the analytic direction] is that of the law giving the rate of decrease of the best approximation of a function as the degree of the approximating polynomial is increased. Research by* Lebesgue, Vallée Poussin, Jackson, *and by me [i.e. Bernstein] has solved this problem in its essential features.*

Bernstein also raises the question:

> *How can this be explained? [...] The reason lies deeper, in the natural process of development of mathematical ideas, a process which, in the first approximation, can be characterized by a brief formula: from the finite to the infinite, from equations to inequalities, from algebra to analysis.*

In regard to the step from the finite to the infinite, Bernstein[10] adds:

> Chebyshev *himself is a less orthodox follower of his own school than are his most immediate students, and he is not entirely a stranger to the direction we have called*

[10] The authors have not had a chance to check Bernstein's more recent evaluation [24].

> *analytic. The fact is that his interest in the theory of mechanisms caused Chebyshev to pose problems which cannot be solved by algebra, and he was the first to attempt a more or less general method for the approximate calculation of the best approximation.*

As to Chebyshev's students, S. N. Bernstein regards E. I. Zolotarev and the brothers A. A. and V. A. Markov as "the most prominent." Further,

> *The above-mentioned start made by* Chebyshev, *which moved him in the direction of an analytic investigation [...] was not taken up by any of his students [...]. With the death of* Chebyshev, *however, his students ceased their investigations in his favorite field, and during the last twenty years [i.e., 1894–1913] there has not appeared a single important work in the algebraic direction.*

These quotations contain some harsh words. But perhaps there is some truth to them.

Here it may appropiate to mention that A. A. Markov, sometimes regarded as Chebyshev's successor, treated in his thesis not only the Markov–Stieltjes inequalities (see Szegö [86]) but also discovered there the polynomials orthogonal on a finite geometric progression for the measure that puts mass q^k at q^k on the points $1, q, q^2, ..., q^n$. This is a second discrete extension of Legendre polynomials, at the same level as Chebyshev's version on an arithmetic progression, but on a geometric progression. The q-version of Chebyshev's extension of Jacobi polynomials had to wait for Hahn's work in the 1940s [87].

Chebyshev began his work in pure mathematics, namely in probability theory, specifically with Poisson's law of large numbers (1846), and in number theory, with an improvement of Legendre's approximate formula in prime number theory (1849, 1852). After he had proven his great talent in pure mathematics and realized that number theory at the time had no visible practical applications he turned more and more to his more natural inclinations, towards the applications, specifically approximation theory and the construction of mechanisms. Over a dozen articles from 1861 to 1888 are in fact devoted to his many technological inventions. This transformation is associated with his European tour of 1852 and his appointment to the chair of applied mathematics at the Academy in 1853. In this respect one has to recall that Chebyshev not only grew up in the tradition leading back to Euler—who had spent 35 years in St. Petersburg (from 1727 to 1741, from 1766 to 1783)—but he also became very familiar with Euler's work since he worked on the new edition of Euler's papers on number theory under Buniakovskii, soon after he moved to St. Petersburg, work undertaken by the Academy.

Even more is true, Chebyshev and Euler as mathematicians had much in common. Both were interested in a great variety of problems, from number theory to engineering. Both were fully aware of the so necessary intercommunication between theory and applications. Both sought the most effective solutions of problems and the discovery of algorithms giving either an exact numerical answer or at least a good approximation. Thus Chebyshev's work was in the spirit of the mathematics of the eighteenth and the first half of the nineteenth centuries, that of Euler, Lagrange, and Poncelet, but not that of Gauss and Cauchy, who introduced especially complex function theory into mathematics, nor that of Riemann or Weierstrass. For Chebyshev, "the partisans of Riemann's extremely abstract ideas delve ever deeper into function-theoretic research and pseudo-geometric investigations in spaces of four and more dimensions". Chebyshev "always remained on solid ground... ." Further, mathematical physics and its general methods did not interest him.

In regard to the transition from equations to inequalities, Chebyshev was a central representative of the *mathematics of inequalities* of the second half of the nineteenth century. A typical example is the Bienaymé–Chebyshev inequality.

Let us close this paper with a quotation from P. J. Davis [34, pp. 14, 119]. He argues that Chebyshev "is one of the patron saints of Russian mathematics, and it is due in no small measure to him that Russian mathematics today [i.e., in 1983][11] stands second to none in the world." Further, also citing Norman Levinson, Davis believes that the Russian success in space travel during the 1950s can be attributed to mathematicians of the Russian scientific establishment, intellectual great–grandsons of Chebyshev.

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[11] Concerning the situation of mathematics in Russia today, see [95] and the literature cited there (see also [88]). The slow decline began to set in already in the fall of 1968.

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